A Poset of Graphs

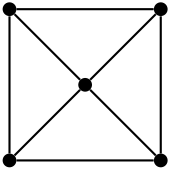
Jason P. Smith

University of Aberdeen

The Poset

$\mathcal{P} =$ Poset of all unlabelled finite graphs, upto isomorphism

Loops and multiple edges ARE allowed.



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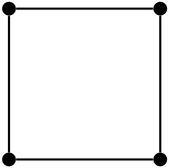
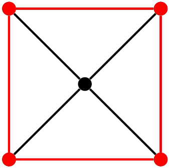


$\leq$



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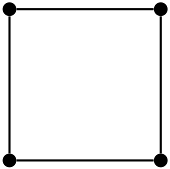
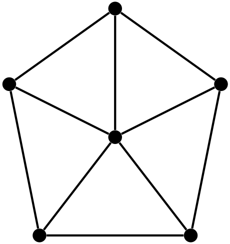


$\not\leq$



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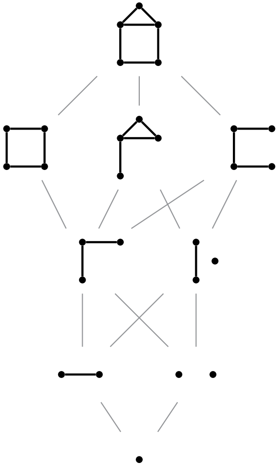

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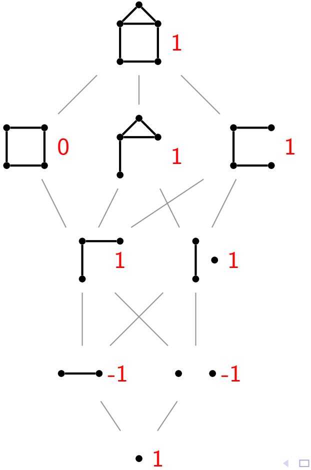
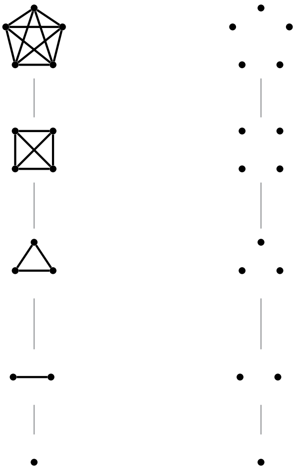


$\mu(\bullet,$  $) = 1$

Complete/Empty Graph



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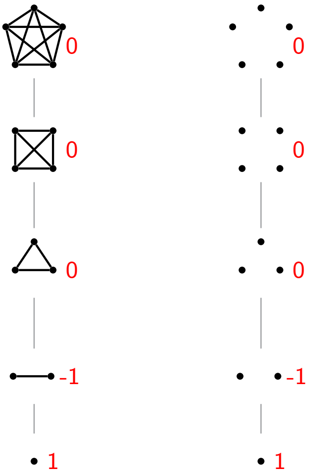

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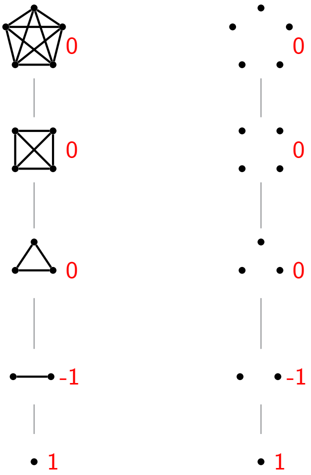
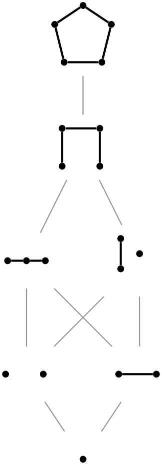


$$\left.\begin{array}{r}\mu(K_a, K_b)\\ \mu(\overline{K_a}, \overline{K_b})\end{array}\right\} = 0$$

$b > a + 1$

Cycle Graph



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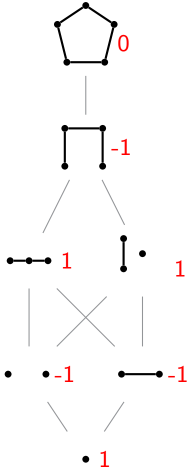

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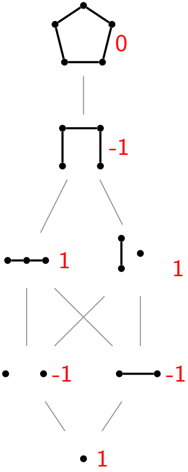

$\mu(H, C_a) = 0$

$H \not\in \{P_{a-1}, C_a\}$

First Results

Vertex Transitive: Deleting any vertex gives same graph, upto isomorphism

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Lemma

If G is vertex transitive, then $\mu(H, G) = 0$ $(\forall H \neq G, G - \{x\})$

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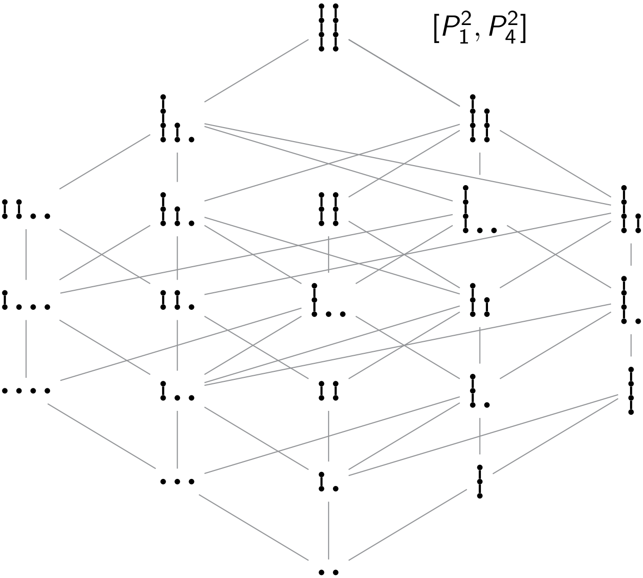
If H, G simple graphs, then $\mu(H, G) = \mu(\overline{H}, \overline{G})$

Lemma

If G has no loops, then $\mu(K_0, G) = 0$ $(|G| > 1)$

Path Graphs



Path Graphs - $\mu(P_1^n, P_x^n)$

$\frac{x\rightarrow}{n\downarrow}$	1	2	3	4	5	6	7	8	9
1	1	-1	1	-1	1	-1	1	-1	1
2	1	0	1	1	2	3	5	8	
3	1	0	1	-1	5	-14	47		
4	1	0	1	1	14	81			
5	1	0	1	-1	42				
6	1	0	1	1					
7	1	0	1						
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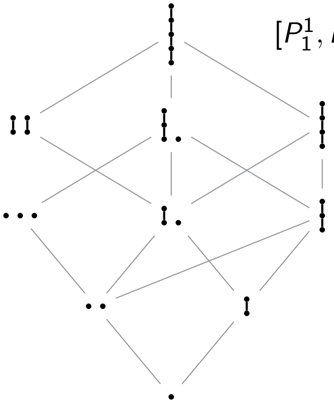
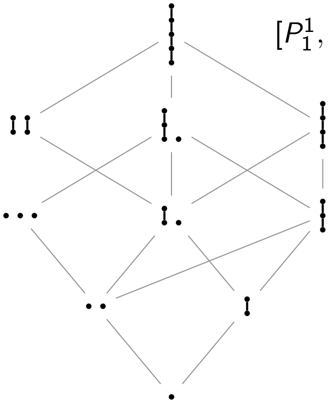
9	1								

Path Graphs - $\mu(P_1^n, P_5^n)$



$[P_1^1, P_5^1]$

Path Graphs - $\mu(P_1^n, P_5^n)$



$[P_1^1, P_5^1]$

$P_5 \to P_2, P_2 \qquad P_5 \to P_4 \qquad P_4 \to P_2, P_1 \qquad P_4 \to P_3 \qquad P_3 \to P_2$

$P_3 \to P_1, P_1 \qquad P_5 \to P_3, P_1 \qquad P_1 \to P_0 \qquad P_2 \to P_1$

Path Graphs - $\mu(P_1^n, P_5^n)$

$$\mu(P_1^n, P_5^n) = \#\text{critical chains from } P_5^n \text{ to } P_1^n$$

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$P_5 > P_{2,2} > P_{2,1} > P_2 > P_1$	:
$P_5 > P_{2,2} > P_{2,1} > P_{1,1} > P_1$	: $P_{1,1}$
$P_5 > P_4 > P_{2,1} > P_2 > P_1$	: P_4
$P_5 > P_4 > P_{2,1} > P_{1,1} > P_1$	: $P_4, P_{1,1}$
$P_5 > P_4 > P_3 > P_2 > P_1$	: P_3
$P_5 > P_4 > P_3 > P_{1,1} > P_1$	: $P_3, P_{1,1}$
$P_5 > P_{3,1} > P_{2,1} > P_2 > P_1$	: $P_{3,1}$
$P_5 > P_{3,1} > P_{2,1} > P_{1,1} > P_1$	: $P_{3,1}, P_{1,1}$
$P_5 > P_{3,1} > P_{1,1,1} > P_{1,1} > P_1$	: $P_{1,1,1}$
$P_5 > P_{3,1} > P_3 > P_2 > P_1$	: $P_{3,1}, P_3$
$P_5 > P_{3,1} > P_3 > P_{1,1} > P_1$	: $P_{3,1}, P_3, P_{1,1}(CC)$

Path Graphs - $\mu(P_1^n, P_5^n)$

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$P_5 > P_{2,2} > P_{2,1} > P_2 > P_1$	:
$P_5 > P_{2,2} > P_{2,1} > P_{1,1} > P_1$	: $P_{1,1}$
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$P_5 > P_4 > P_{2,1} > P_{1,1} > P_1$	: $P_4, P_{1,1}$
$P_5 > P_4 > P_3 > P_2 > P_1$	: P_3
$P_5 > P_4 > P_3 > P_{1,1} > P_1$	: $P_3, P_{1,1}$
$P_5 > P_{3,1} > P_{2,1} > P_2 > P_1$	: $P_{3,1}$
$P_5 > P_{3,1} > P_{2,1} > P_{1,1} > P_1$	: $P_{3,1}, P_{1,1}$
$P_5 > P_{3,1} > P_{1,1,1} > P_{1,1} > P_1$	: $P_{1,1,1}$
$P_5 > P_{3,1} > P_3 > P_2 > P_1$	: $P_{3,1}, P_3$
$P_5 > P_{3,1} > P_3 > P_{1,1} > P_1$	: $P_{3,1}, P_3, P_{1,1}(CC)$

Critical chains: $P_5 > P_{3,1} > P_3$ $P_3 > P_{1,1} > P_1$

Path Graphs - $\mu(P_1^n, P_5^n)$

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$P_5 > P_{2,2} > P_{2,1} > P_2 > P_1$	:
$P_5 > P_{2,2} > P_{2,1} > P_{1,1} > P_1$	: $P_{1,1}$
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$P_5 > P_4 > P_{2,1} > P_{1,1} > P_1$	: $P_4, P_{1,1}$
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$P_5 > P_{3,1} > P_{1,1,1} > P_{1,1} > P_1$	: $P_{1,1,1}$
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$P_5 > P_{3,1} > P_3 > P_{1,1} > P_1$	: $P_{3,1}, P_3, P_{1,1}(CC)$

Critical chains: $P_5 > \underset{U}{P_{3,1}} > P_3$ $P_3 > \underset{D}{P_{1,1}} > P_1$

Path Graphs - $\mu(P_1^n, P_5^n)$

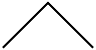
Critical chains: $\underset{U}{P_5 > P_{3,1} > P_3}$ $\underset{D}{P_3 > P_{1,1} > P_1}$

$P_5 > P_{3,1} > P_3 > P_{1,1} > P_1$ 

Path Graphs - $\mu(P_1^n, P_5^n)$

Critical chains: $\underset{U}{P_5 > P_{3,1} > P_3}$ $\quad$ $\underset{D}{P_3 > P_{1,1} > P_1}$

$P_5 > P_{3,1} > P_3 > P_{1,1} > P_1$ 

$P_{5,5} > P_{5,3,1} > P_{5,3} > P_{3,3,1} > P_{3,3} > P_{3,1,1} > P_{3,1} > P_{1,1,1} > P_{1,1}$ 

$P_{5,5} > P_{5,3,1} > P_{5,3} > P_{5,1,1} > P_{5,1} > P_{3,1,1} > P_{3,1} > P_{1,1,1} > P_{1,1}$ 

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		✓	✓	✓	✓	✓				
	$\frac{x\rightarrow}{n\downarrow}$	1	2	3	4	5	6	7	8	9
✓induction	1	1	-1	1	-1	1	-1	1	-1	1
✓induction	2	1	0	1	1	2	3	5	8	
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Path Graphs - $\mu(P_1^n, P_x^n)$

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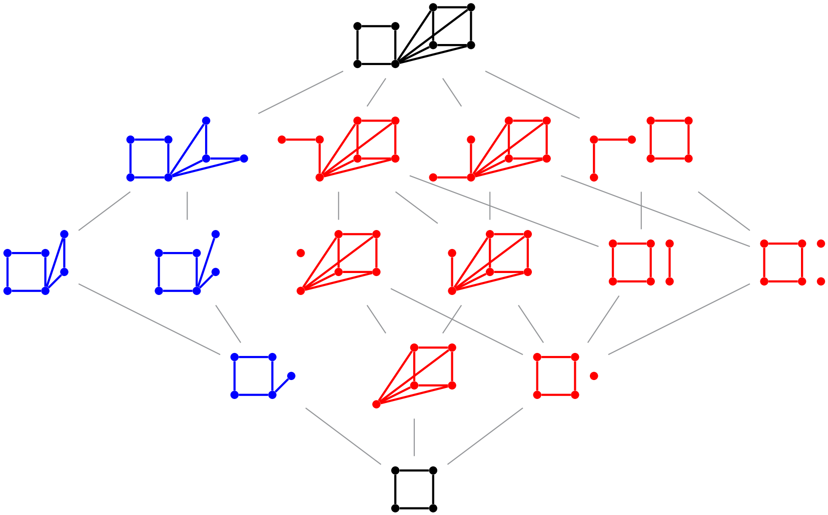
Corollary

The Möbius function is unbounded.

Disconnected Intervals

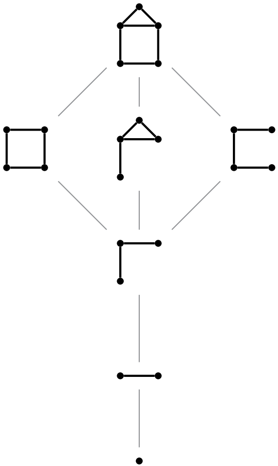


Theorem

[H, G] is disconnected if and only if it is strongly zero-split.

Poset of Connected Graphs

$\mathcal{P}^c =$ Poset of all unlabelled finite connected graphs



Poset of Connected Graphs

$\mathcal{P}^c =$ Poset of all unlabelled finite connected graphs

- $\mu(K_1, G) = 0$
- Möbius function is unbounded
- $[H, G]$ disconnected if and only if it strongly zero split
- $\mu(K_a, K_b) = \mu(\overline{K_a}, \overline{K_b}) = \mu(H, C_a) = \mu(P_a, P_b) = 0$

Further Work

- Are the coatoms of every interval unique? (Graph Reconstruction Conjecture)
- What intervals are shellable? Proportion? Disconnected subinterval necessary and sufficient?
- Unimodal?
- Proportion of intervals with nonzero Möbius function?
- Poset of directed graphs?

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- Poset of directed graphs?

Thank You For Listening!

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