Combinatorial Algebraic Topology and its Applications to Permutation Patterns

Jason P Smith

University of Strathclyde

Overview

1. Introduction to Combinatorial Algebraic Topology
   - Basic Topology
   - Graphs to Simplicial Complexes
   - Posets to Simplicial Complexes

2. Permutation Patterns
   - Introduction and Motivation
   - Applying Combinatorial Algebraic Topology

Kozlov, Dimitry. *Combinatorial algebraic topology*. Vol. 21. Springer Science & Business Media, 2008.

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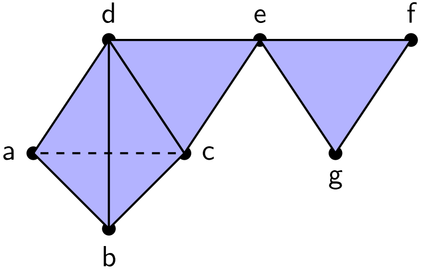

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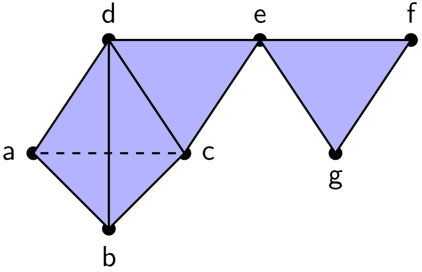

$\dim \Delta = 3$ and non-pure

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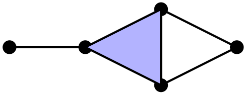

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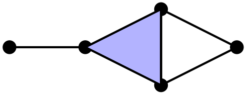

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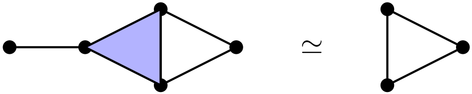

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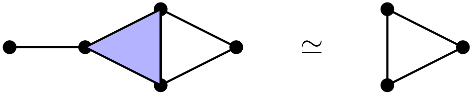
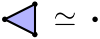


Contractible $\implies$ homotopy equivalent to a point: 

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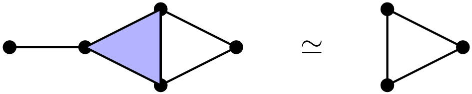
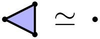


Contractible $\implies$ homotopy equivalent to a point: 

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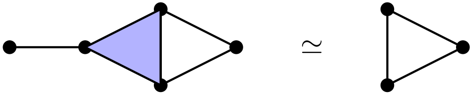
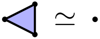


Contractible $\implies$ homotopy equivalent to a point: 

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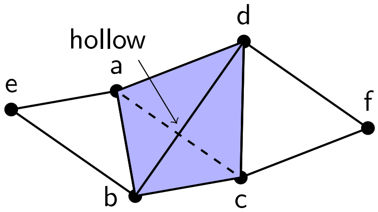

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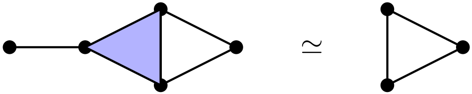
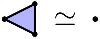


Contractible $\implies$ homotopy equivalent to a point: 

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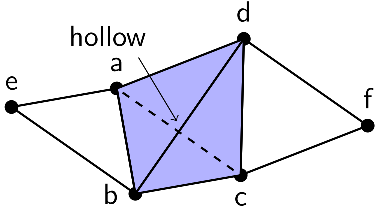


$\tilde{\beta}_{-1} = 0$

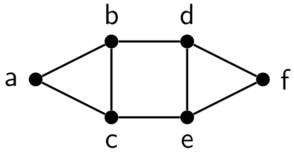
$\tilde{\beta}_0 = 0$

$\tilde{\beta}_1 = 2$

$\tilde{\beta}_2 = 1$

$\tilde{\chi} = -1$

Graphs and the Colouring Problem



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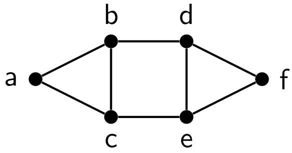

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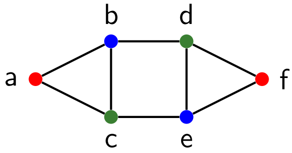

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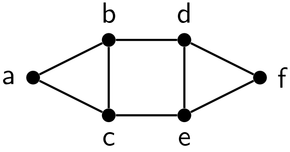
Chromatic number $\chi(G) = 3$

Graphs to Simplicial Complexes



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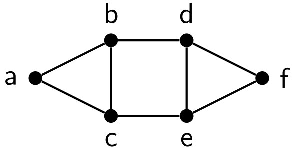
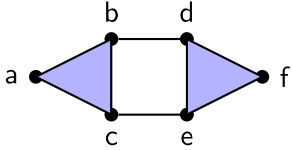


Flag/Clique Complex $\mathrm{Cl}(\mathrm{G})$



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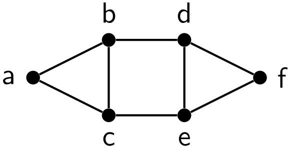

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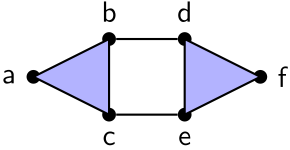
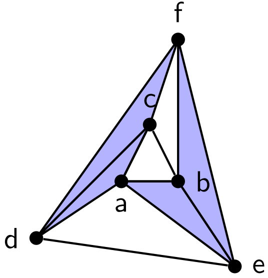


Neighbourhood Complex $\mathcal{N}(G)$



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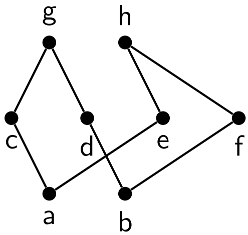

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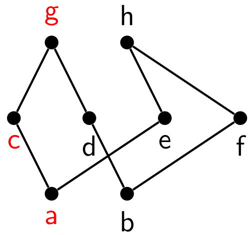

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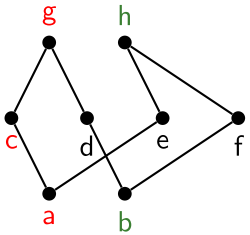

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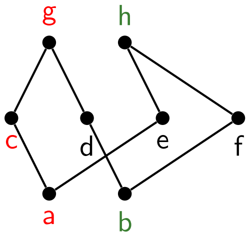

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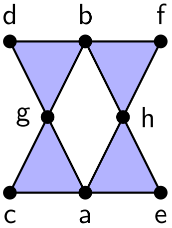

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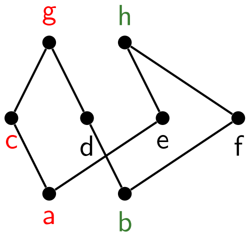

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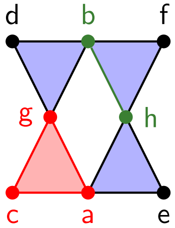

Chains of a poset P give faces of the *Order Complex* $\Delta(P)$.

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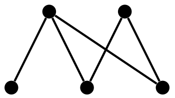
$P =$ 

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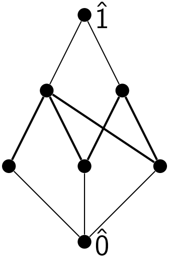

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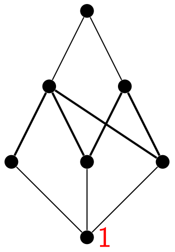

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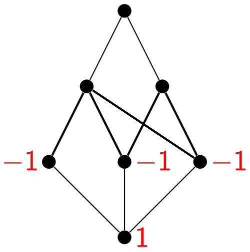

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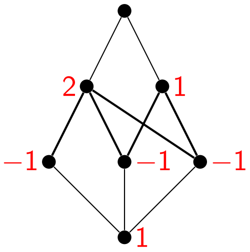

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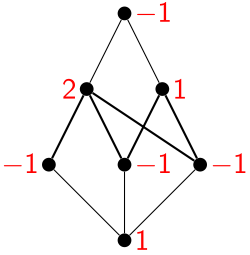

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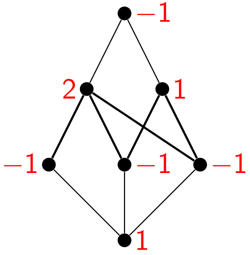

$\mu(P) = -1$

Applications

Lemma

$$\mu(P) = \tilde{\chi}(\Delta(P))$$

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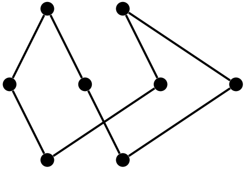

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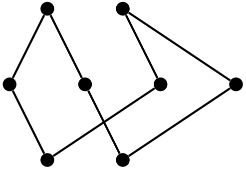
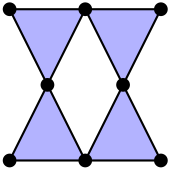


$$\downarrow \Delta$$



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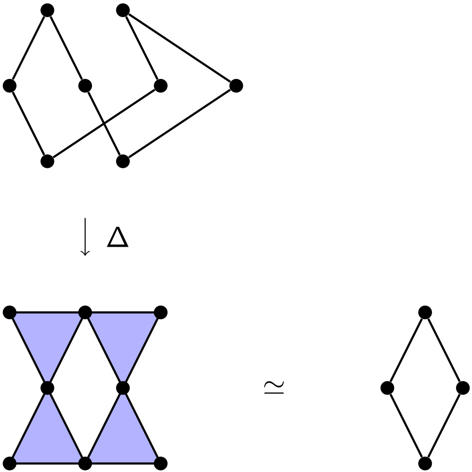

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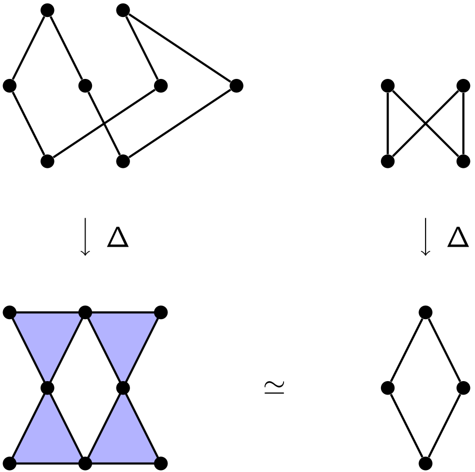

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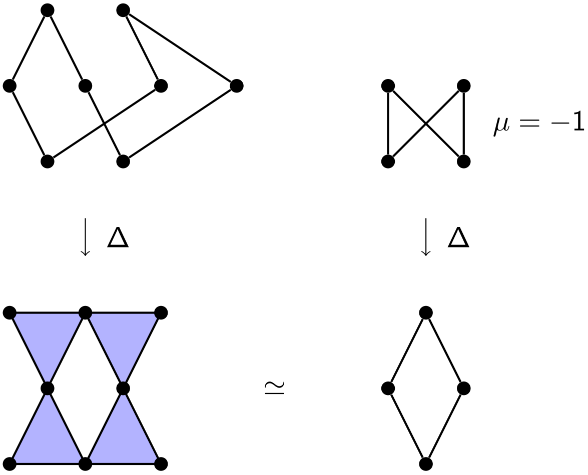

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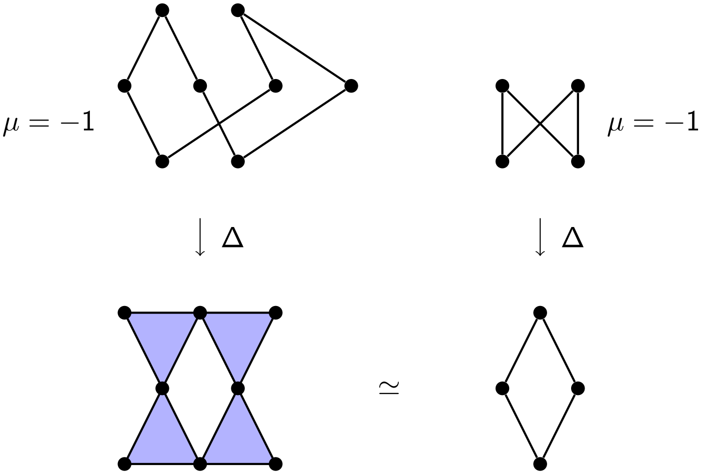

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Let $[n] := \{1, \dots, n\}$ and $X^k := \{x \subseteq [n] \mid |x| = k\}$ compute $\displaystyle\sum_{\substack{A \subseteq X^k \\ \cup A = [n]}} (-1)^{|A|}$

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Proposition (Crosscut Theorem)

Consider poset P and subset X s.t $\forall p \in P \, \exists x \in X$ s.t $p \geq x$, then:

$$\mu(\hat{0}, \hat{1}) = \sum_{\substack{A \subseteq X \\ \vee A = \hat{1}}} (-1)^{|A|}.$$

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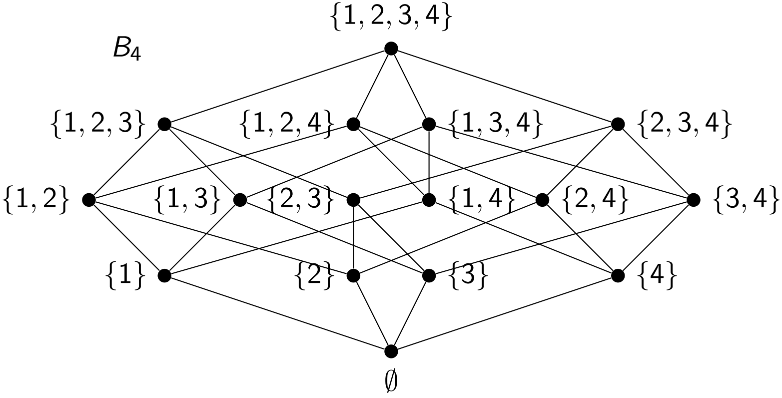

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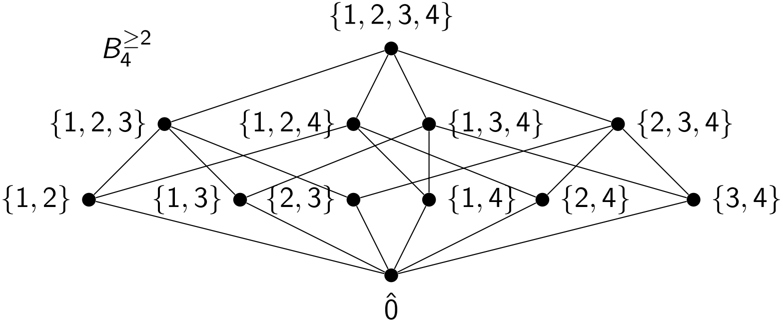

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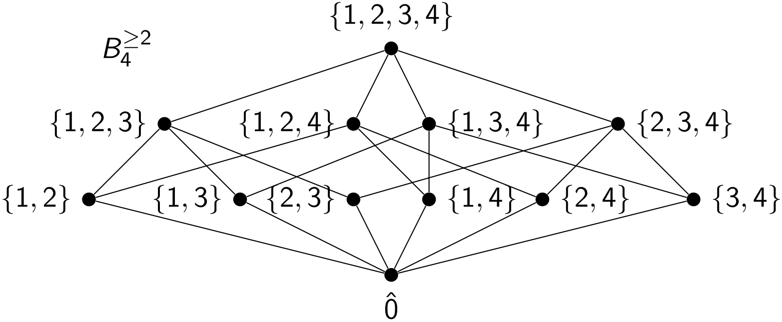

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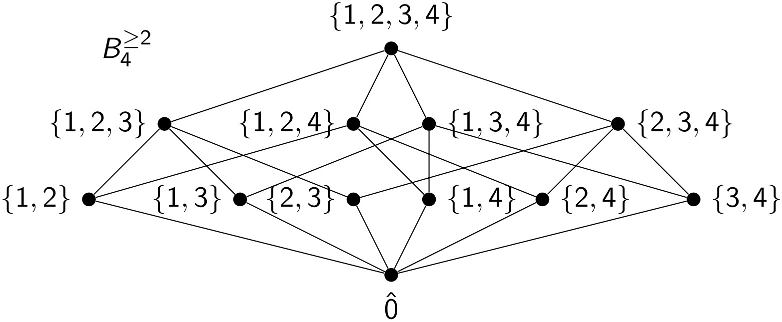

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Single line notation for permutations i.e 241365.

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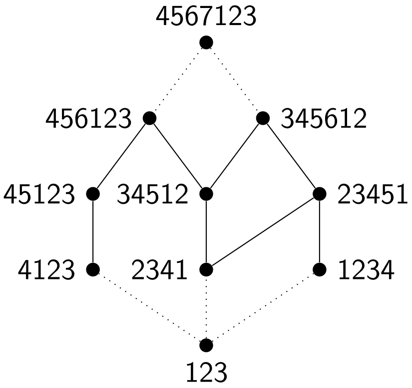
Lots of work in enumerating avoidance of permutations. Studying the Möbius function and topology of $\mathcal{P}$ can help with this.

The Interval $[123, 4567123]$



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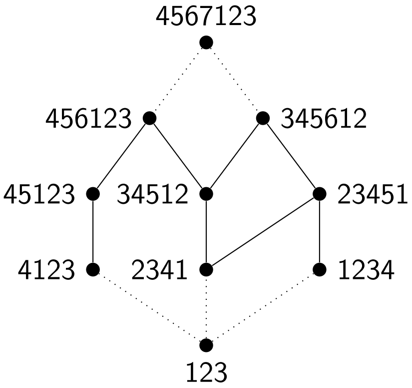

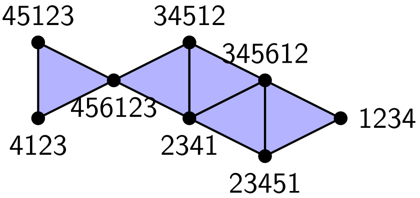

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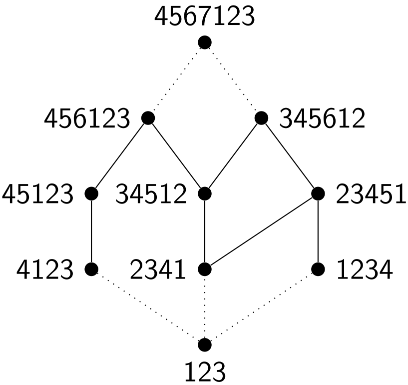

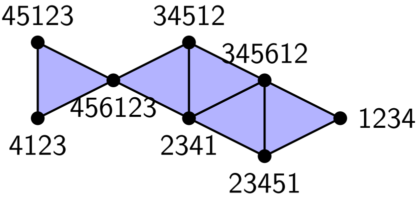

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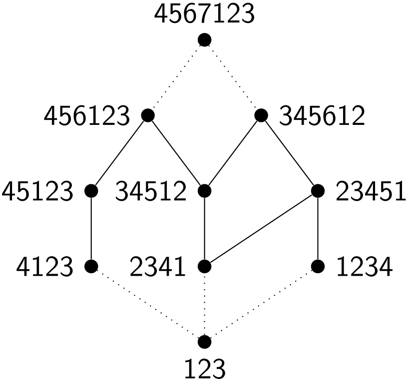

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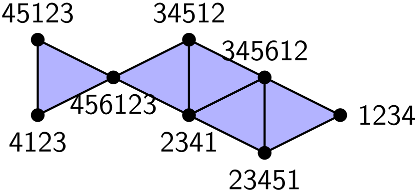

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Very few intervals satisfy these properties. But there is a common theme of *normal embeddings*.

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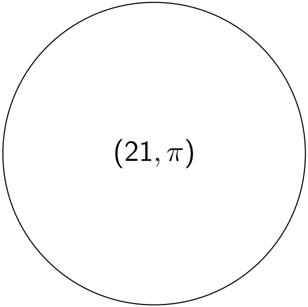

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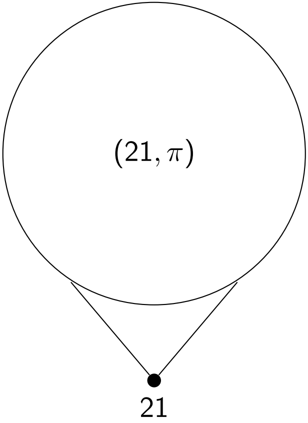

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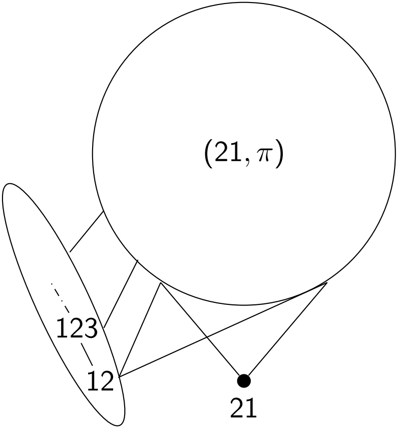

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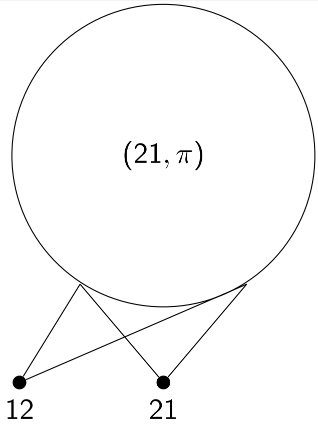

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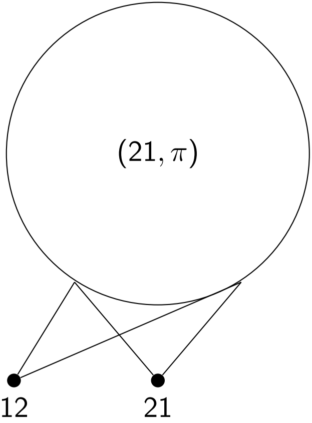

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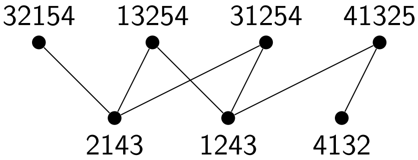
The only normal occurrence is 235478, so $\mathrm{NE}(124356, 23165478) = 1$.

Breakdown Permutation: $[132, 413265]$



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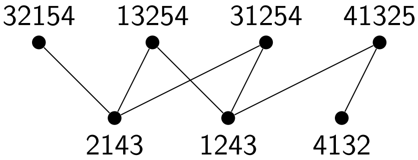

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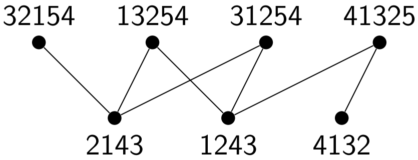

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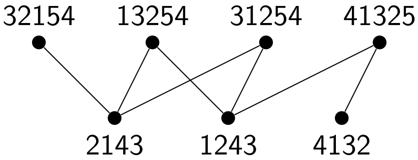

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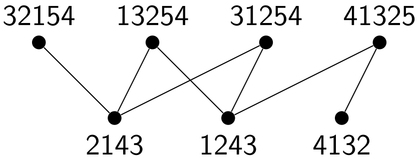

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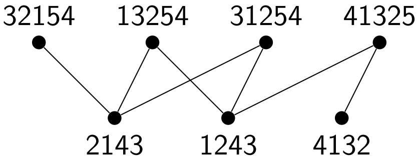

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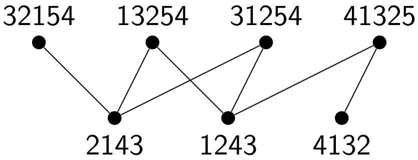

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$$\mu(P(\eta)) = \begin{cases} 0, & \eta \text{ not normal} \\ -1^{|\pi|-|\sigma|}, & \eta \text{ normal} \end{cases}$$

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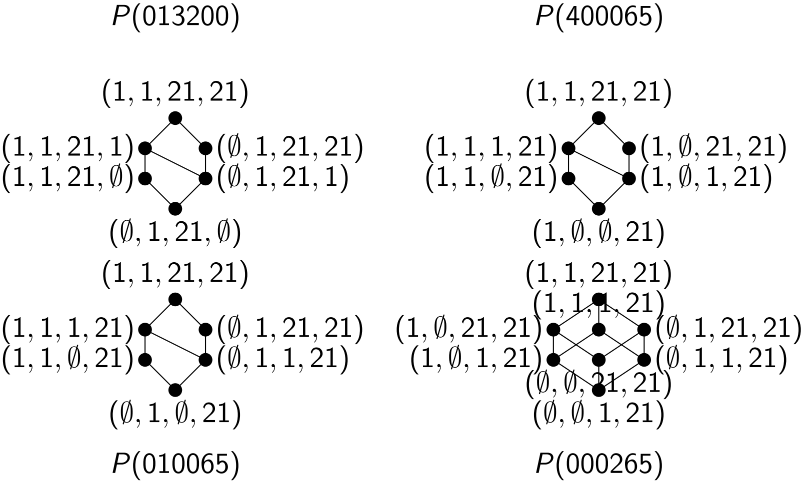

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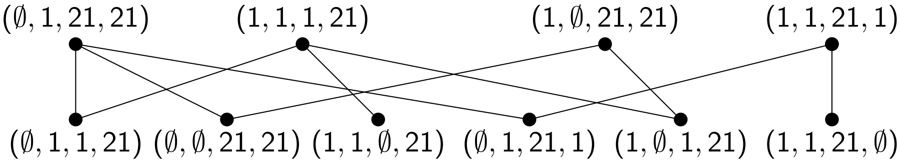
$$A^{\sigma,\pi} := \bigcup_{\eta \in E^{\sigma,\pi}} P(\eta)^o$$



$A^{132,413265}$

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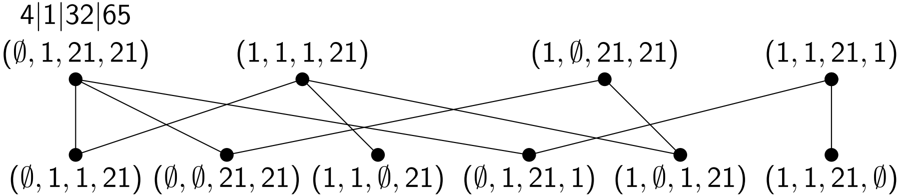

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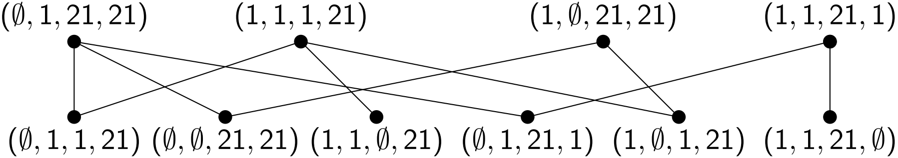

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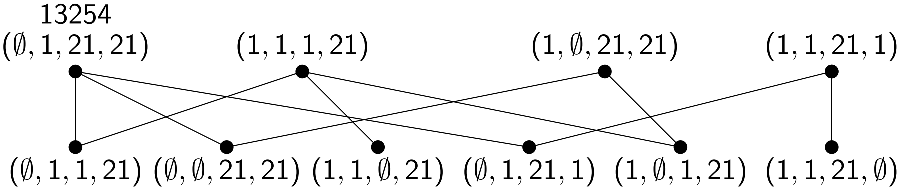

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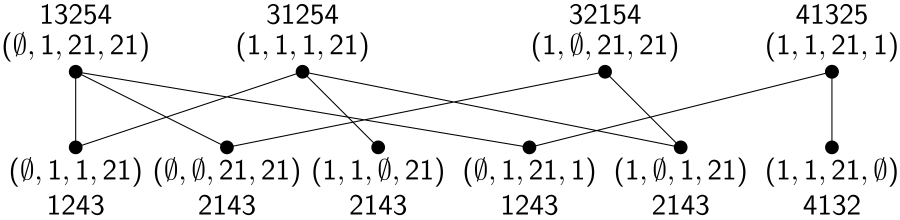

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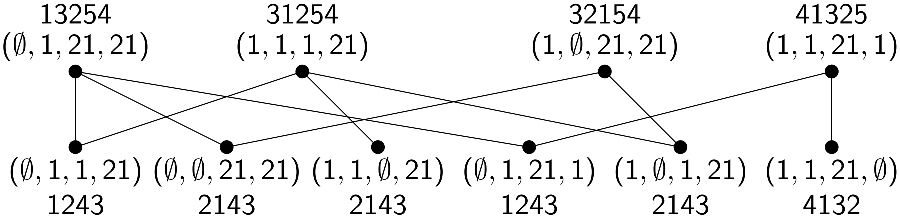

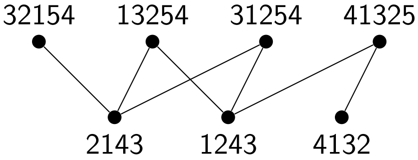

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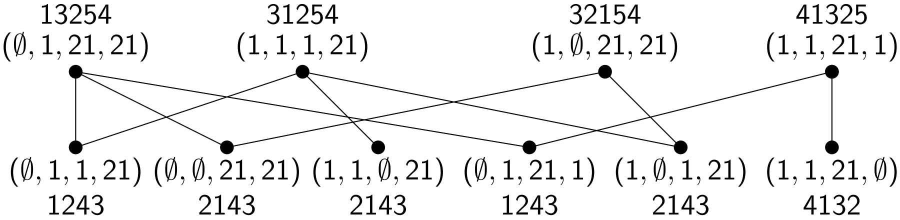

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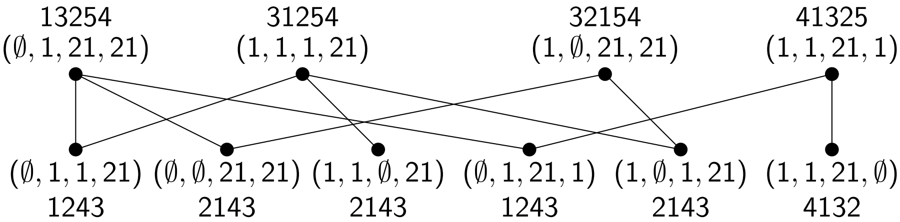

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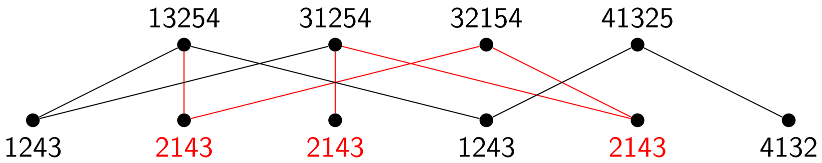

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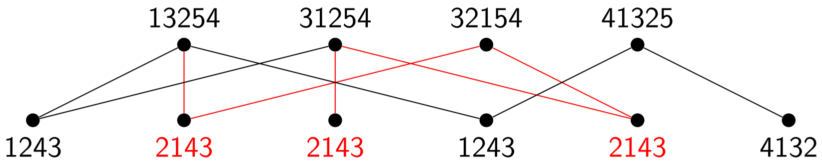

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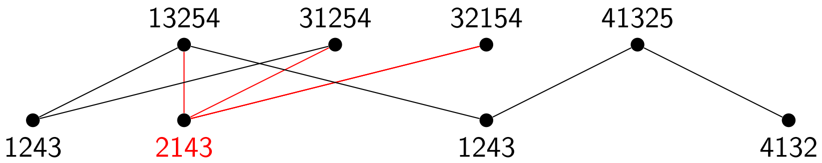

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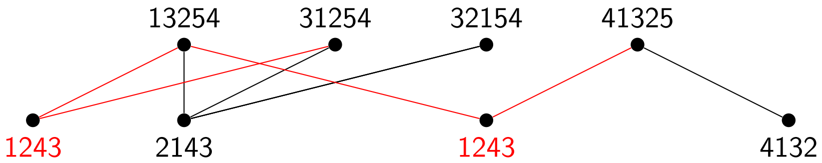

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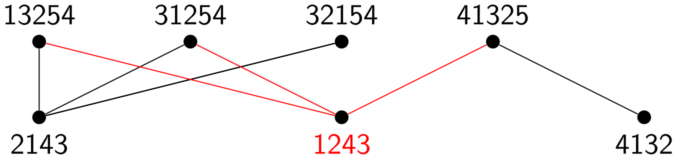

$$\mu(A^{\sigma,\pi}) - \mu(W(\sigma, 2143))\mu(W(2143, \pi))$$
$$- \mu(W(\sigma, 1243))\mu(W(1243, \pi)) = \mu(\sigma, \pi)$$

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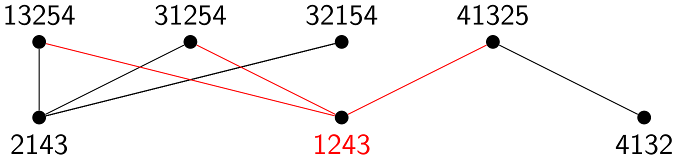

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Thank You For Listening