Intervals of Permutations with a Fixed Number of Descents are Shellable

Jason P Smith

University of Strathclyde

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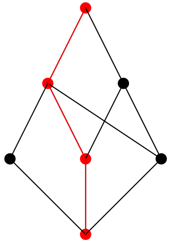
- An *interval* $[\sigma, \pi]$ consists of all permutations $\sigma \leq z \leq \pi$. An *open interval* (σ, π) consists of all permutations $\sigma < z < \pi$.
- A *chain* in a poset is a totally ordered subset. In $[1, 24513]$ there is a chain $21 < 2341 < 24513$ or a different example is:



Simplicial Complexes

Definition

An abstract simplicial complex Δ on a finite vertex set V is a nonempty collection of subsets of V such that:

- $\{v\} \in \Delta$ for all $v \in V$
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The elements of Δ are called *faces*. The maximal faces are called *facets*. The *dimension* of Δ is $\max_{F \in \Delta} |F| - 1$.

Simplicial Complex Example

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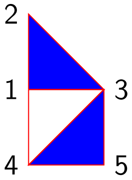

This is a two dimensional simplicial complex where the vertex set is $V = \{1, 2, 3, 4, 5\}$ and

$$\Delta = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{3, 4\}, \{3, 5\}, \{4, 5\}, \{1, 2, 3\}, \{3, 4, 5\}\}.$$

Betti numbers of simplicial complexes

The i'th *Betti number*, β_i^Δ, of a simplicial complex Δ, is the rank of the i'th homology group.

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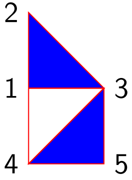
This simplicial complex has Betti numbers $\beta_0 = 0$, $\beta_1 = 1$ and $\beta_2 = 0$:



Order Complexes

The order complex $\Delta(\sigma, \pi)$ of an interval $[\sigma, \pi]$:

- Vertices of $\Delta(\sigma, \pi)$ are the elements of (σ, π).
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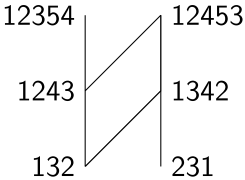

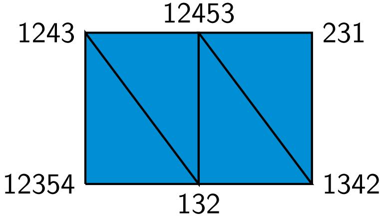

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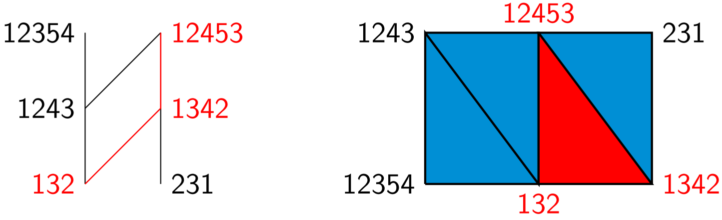

Shellability

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are pure and $(\dim \Delta - 1)$-dimensional, for $k = 2, \ldots, t$. Such an ordering of the facets is called a *shelling*.

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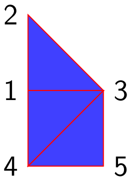
  are pure and $(\dim \Delta - 1)$-dimensional, for $k = 2, \ldots, t$.
  Such an ordering of the facets is called a *shelling*.
- A shellable simplicial complex Δ has $\beta_i^\Delta = 0$ for any $i < \dim \Delta$.

Shellable Example



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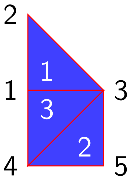

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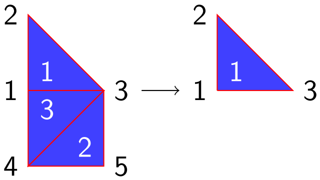

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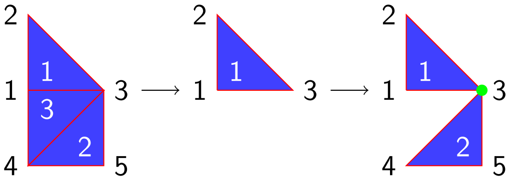

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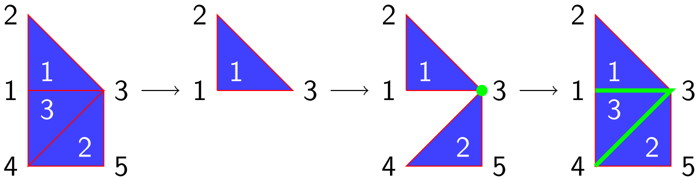

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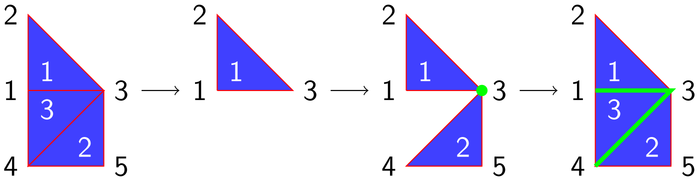

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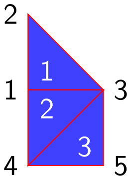

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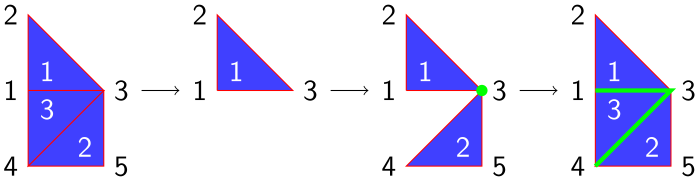

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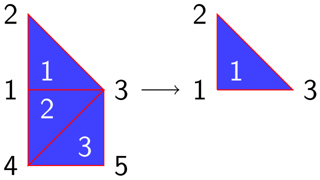

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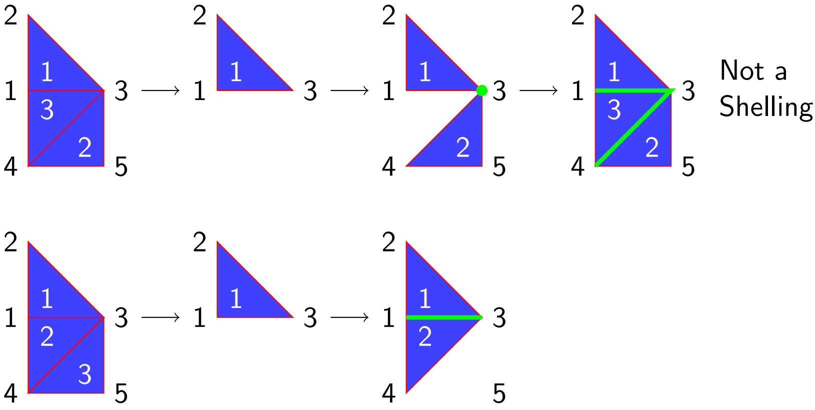

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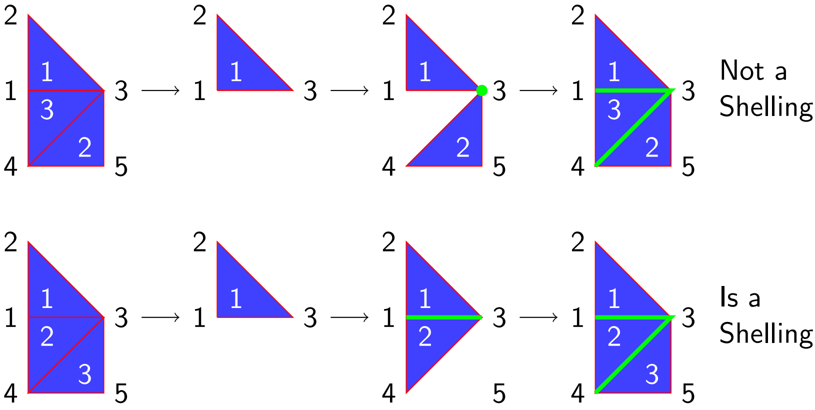

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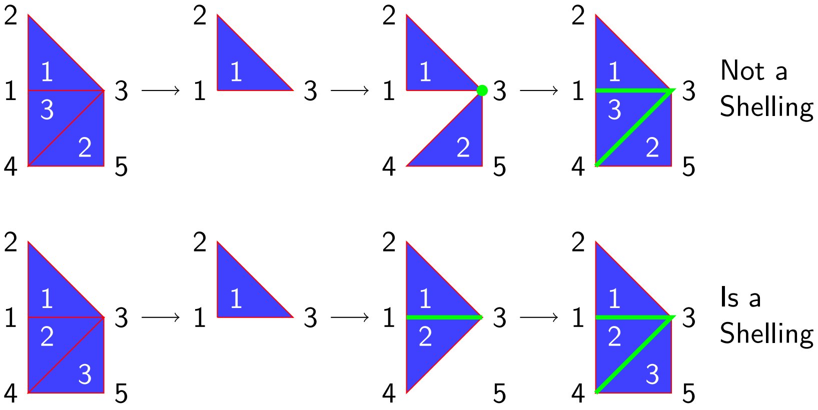

This Complex is Shellable

Background

- (Björner, 1980): An interval that contains a non-trivial disconnected subinterval is not shellable.
- (McNamara and Steingrímsson, 2013): Intervals of layered permutations, that do not contain a non-trivial disconnected subinterval, are shellable.
- (Billera and Myers, 2006): A (2+2)-free poset is shellable.
- (Björner, 1988): Any interval of words with subword order is shellable and a formula for the Möbius function is presented.

Shellability of Intervals of Permutations with a Fixed Number of Descents

Theorem

If two permutations σ and π have the same number of descents then $[\sigma, \pi]$ is shellable.

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Subword order: $v \leq w$ if there is a subsequence $w_{i_1} \dots w_{i_m}$ in w such that $v = w_{i_1} \dots w_{i_m}$. $2132 \leq 212312$ but $2132 \not\leq 21233$.

The Poset of Words

Definition

Let $\widehat{\mathcal{A}}$ denote the poset of words with subword order on the alphabet of all positive integers with the additional conditions that for any $w \in \widehat{\mathcal{A}}$:

AC1: There is at least one occurrence of each letter $i \in \{1, \ldots, \max(w)\}$.

AC2: The rightmost occurrence of each letter $i \in \{1, \ldots, \max(w) - 1\}$ is preceded by an occurrence of $i + 1$.

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Let $\widehat{\mathcal{A}_k}$ be the subposet of $\widehat{\mathcal{A}}$ of words w where $\max(w) = k$.

For example $231243 \in \widehat{\mathcal{A}}$ but $21343 \notin \widehat{\mathcal{A}}$.

Order Isomorphism

Let $d_\pi(c)$ be the number of descents preceding the letter c in π plus 1:

$$23154$$
$$11223$$

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We define the order isomorphism:

$$f : \mathcal{P}_k \to \widehat{\mathcal{A}_{k+1}} : \pi \mapsto d_\pi(1)d_\pi(2)\dots d_\pi(|\pi|),$$

$$f^{-1} : \widehat{\mathcal{A}_{k+1}} \to \mathcal{P}_k : w \mapsto [i : w_i = 1][i : w_i = 2]\dots[i : w_i = k+1].$$

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$$f(263415) = 3122$$

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$$f^{-1}(214321) =$$

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$$f(263415) = 312231$$

$$f^{-1}(214321) = 26$$

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$$f(263415) = 312231$$

$$f^{-1}(21{\color{red}4}321) = 261543$$

Möbius function

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The *Möbius function* for a poset is defined as $\mu(a, b) = 0$ if $a \not\leq b$, $\mu(a, a) = 1$ for all a and for $a < b$:

$$\mu(a, b) = -\sum_{a \leq z < b} \mu(a, z).$$

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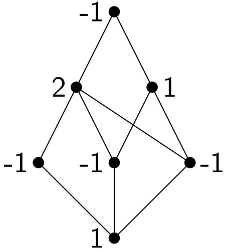

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$$\mu(\sigma,\pi) = \sum_i (-1)^i \beta_i^{\Delta(\sigma,\pi)}$$

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3567124

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The normal occurrences of 3412 in 3567124 are 6712 and 6724.

Denote the number of normal occurrences of σ in π as $\binom{\pi}{\sigma}_n$.

$$\binom{3567124}{3412}_n = 2$$

The Möbius function of Intervals of Permutations with a Fixed Number of Descents

Theorem

If σ and π are permutations with the same number of descents, then

$$\mu(\sigma, \pi) = (-1)^{|\pi|-|\sigma|} \binom{\pi}{\sigma}_n .$$

Other Results

Lemma

If σ and π have the same number of descents then $\Delta(\sigma, \pi)$ is homotopy equivalent to a wedge of $|\mu(\sigma, \pi)|$ spheres of dimension $\dim(\Delta(\sigma, \pi))$.

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Corollary

If π has exactly one descent then $\mu(1, \pi) = -\mu(21, \pi)$.

Möbius Function

Theorem

Let π be of length $n > 2$ with exactly one descent at position d:

1. *If $\mu(1, \pi) \neq 0$ then $\mu(1, \pi)$ is positive if and only if n is odd.*
2. *If π has exactly two adjacencies and the first adjacency has greater value than the second then $\mu(1, \pi) = \pm 1$,*
3. *If π has exactly one adjacency at position $i < d$ and $\pi_1 \neq 1$ then $\mu(1, \pi) = \pm i$,*
4. *If n is even and $\pi = 135 \ldots (n-1)246 \ldots n$ then $\mu(1, \pi) = -\binom{\frac{n}{2}}{2}$,*

Shellability

Conjecture

If $\pi \in P_1$ and π avoids 456123 and 356124 the interval $[1, \pi]$ is shellable.

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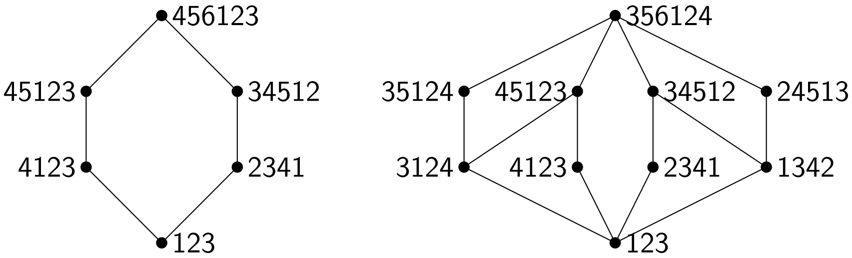

Further Work

1. Consider intervals between permutations whose number of descents differ by one.
2. Consider intervals with a different fixed statistic.
3. When does $\Delta(\sigma, \pi)$ have homotopy type a wedge of spheres and when is it shellable.
4. Other results for the Möbius function through from the homology of these order complexes.

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Thank You For Listening.

http://arxiv.org/abs/1405.2560