A Formula for the Möbius Function of the Permutation Poset

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Abstract

We present a formula for the Möbius function of the poset of permutations, ordered by pattern containment. Computing the Möbius function recursively takes exponential time; our formula splits the computation into two parts, where the first is polynomial and the second exponential. Computer tests indicate the second part vanishes for a significant proportion of intervals. We give a condition under which computing the second term is polynomial and which implies shellability.

The Permutation Poset

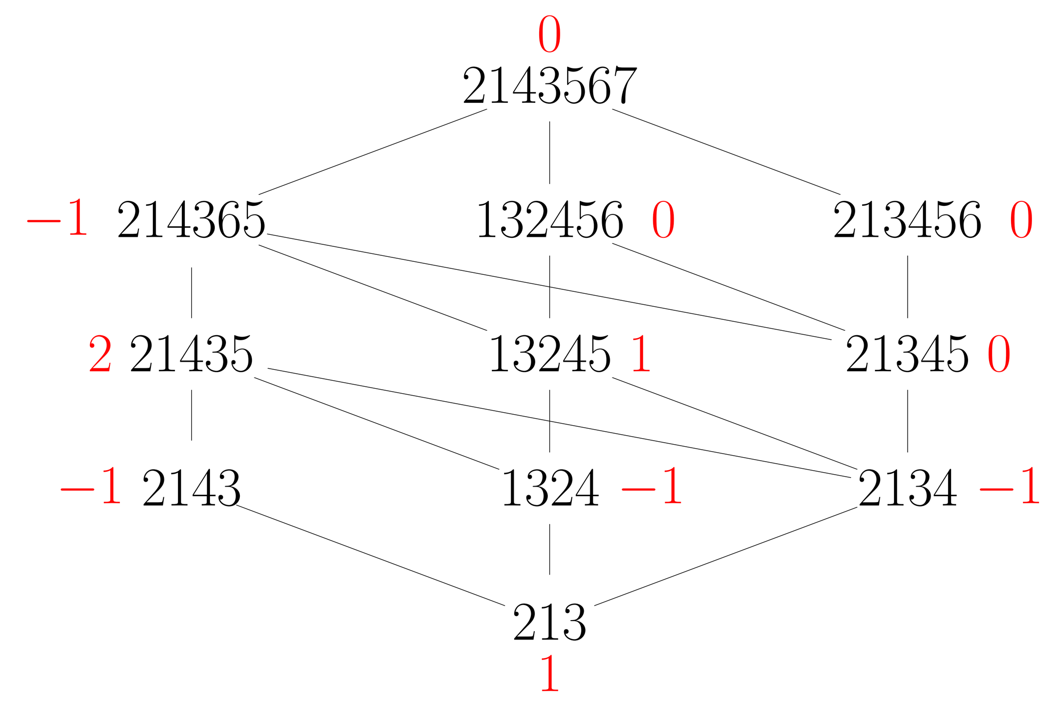
A *permutation* π is an ordering of the letters $1,\dots,n$. Let $\sigma \leq \pi$ if σ *occurs as a pattern* in π, that is, if there is a subsequence of π whose letters appear in the same order of size as the letters of σ. The permutation poset is the set of all permutations with this partial order.

In 25143, 254 is an occurrence of 132, so $132 \leq 25143$

The *Möbius function* for an interval $[a,b]$ of a poset is defined by: $\mu(a,a)=1$ and

$$\mu(a,b) = -\sum_{a\leq z<b} \mu(a,z).$$

A poset is *shellable* if its maximal totally ordered subsets (*chains*) can be ordered in a "nice" way. Shellability implies (many) nice properties of the topology of the poset's order complex.



The interval $[213, 2143567]$ and Möbius function in red.

Generalisation to Pattern Posets

Pattern occurrence has been studied on many other combinatorial objects, such as words, ascent sequences and Dyck paths. In many of these cases we can define a pattern poset using occurrence as a partial order. Many of the results here can be generalised to any pattern poset.

Definitions and Example: $[213, 21435]$

An *adjacency* is a maximal increasing/decreasing sequence of consecutive letters and the *tail* is all but the first letter.

$Adj(21435) = \{21, 43, 5\}$ $\qquad$ $Tails(21435) = \{1, 3\}$

An *interval block* is a sequence of consecutive letters that contains all values between two integers i and $i+a$, $a>0$. An interval block is *free* if it contains no adjacencies of length greater than 1.

$IntBl(21435) = \{21, 2143, 435, 21435\}$ $\qquad$ $FreeIntBl(21435) = \emptyset$

We can obtain λ from π by *shrinking* an interval block, if removing all but the smallest letter of the interval block of π gives λ. $B(\sigma,\pi)$ is the set of elements of $[\sigma,\pi)$ that can be obtained from π by shrinking a free interval block.

Shrinking 435 in 21435 gives 213 $\qquad$ $B(213, 21435) = \emptyset$

An *embedding* is an occurrence with zeroes in the other positions. An embedding is *normal* if all positions in tails are nonzero and *representative* if each adjacency has nonzero letters on the right and zeroes on the left.

$\mathrm{Emb}(213, 21435) = \{21400, 21030, 21005, 00435\}$ $\qquad$ $\mathrm{NE}(213, 21435) = \{21030\}$

$\mathrm{RepEmb}(213, 21435) = \{21030, 21005, 00435\}$

If $\mathrm{Emb}(\sigma,\pi)$ can be split into two nonempty sets with no position occurring as a zero in both, we say $[\sigma,\pi]$ is *zero split*.

$[213, 21435]$ is zero split, due to the partition $\{21400, 21030, 21005\}$ and $\{00435\}$.

Main Results

The Möbius function for any permutations σ and π is given by the following formula, where $\vee$ denotes the join:

$$\mu(\sigma,\pi) = (-1)^{|\pi|-|\sigma|}\,|\,\mathrm{NE}(\sigma,\pi)| + \sum_{\lambda\in[\sigma,\pi)} \mu(\sigma,\lambda) \sum_{\substack{S\subseteq \mathrm{RepEmb}(\sigma,\pi)\\ \vee S=\pi}} (-1)^{|S|}, \tag{1}$$

For any interval $[\sigma,\pi]$ with no zero split subintervals:

$$[\sigma,\pi] \text{ is shellable,} \tag{2}$$

$$\mu(\sigma,\pi) = (-1)^{|\pi|-|\sigma|}\,|\,\mathrm{NE}(\sigma,\pi)| + \sum_{\lambda\in B(\sigma,\pi)} \mu(\sigma,\lambda)(-1)^{|\pi|-|\lambda|-1}. \tag{3}$$

If $[\sigma,\pi]$ has no zero split subintervals and π contains no free interval blocks:

$$\mu(\sigma,\pi) = (-1)^{|\pi|-|\sigma|}\,|\,\mathrm{NE}(\sigma,\pi)| \tag{4}$$

- Computer tests indicate $\mu(\sigma,\pi) = (-1)^{|\pi|-|\sigma|}|\,\mathrm{NE}(\sigma,\pi)|$ for $\sim 95\%$ of intervals $[\sigma,\pi]$, $|\pi|<9$.
- $\mathrm{NE}(\sigma,\pi)$ and $B(\sigma,\pi)$ can both be computed in polynomial time. Normally $\mu(\sigma,\pi)$ takes exponential time.

Applying the Main Results

$[213, 21435]$: One normal embedding and the second term of Equation (1) is nonzero only for $213 \in [213, 21435]$, with sets $\{21030, 00435\}$, $\{21005, 00435\}$ and $\{21030, 21005, 00435\}$. So Equation (1) gives:

$$\mu(213, 21435) = (-1)^{5-3}(1) + \mu(213,213)((-1)^2 + (-1)^2 + (-1)^3) = 2$$

$[213, 132456]$: No zero split subintervals, normal embeddings or free interval blocks, so shellable and $\mu(213, 132456) = 0$.

$[24135, 24136857]$: No zero split subintervals, 4 normal embeddings and $B(24135, 24136857) = \{24135\}$, so shellable and $\mu(24135, 24136857) = -4 + 1 = -3$.

Proof Sketch

- Let $A(\sigma,\pi)$ and $A^*(\sigma,\pi)$ be the posets of all representative embeddings of λ in π for all $\lambda\in(\sigma,\pi)$ and $\lambda\in[\sigma,\pi)$, respectively.
- Let $f : A(\sigma,\pi) \to (\sigma,\pi)$ map elements of $\mathrm{RepEmb}(\lambda,\pi)$ to λ. Then $A(\sigma,\pi)$ and f form a poset fibration.

To prove the results we consider the Möbius function and topology of $A(\sigma,\pi)$ and the effect f has when applied.

1. We prove a general result on poset fibrations giving
$$\mu(\sigma,\pi) = \mu(A(\sigma,\pi)) + \sum_{\lambda\in(\sigma,\pi)} \mu(\sigma,\lambda)\mu(A^*(\lambda,\pi))$$
2. The Crosscut Theorem gives
$$\mu(A^*(\lambda,\pi)) = \sum_{\substack{S\subseteq \mathrm{RepEmb}(\lambda,\pi)\\ \vee S=\pi}} (-1)^{|S|}$$
3. Let $\hat{1}$ be the top element of $A(\sigma,\pi)$, then we can split $A(\sigma,\pi)$ into parts using the embeddings
$$A(\sigma,\pi) = \bigcup_{\eta\in \mathrm{RepEmb}(\sigma,\pi)} (\eta,\hat{1})$$
4. $[\eta,\hat{1}]$ is isomorphic to a product of chains implying
$$\mu(\eta,\hat{1}) = \begin{cases} (-1)^{|\pi|-|\sigma|} & \text{if } \eta \text{ is normal} \\ 0, & \text{if } \eta \text{ is not normal} \end{cases}$$
5. The intersections are either contractible or empty so
$$\mu\left(\bigcap_{\eta\in S}(\eta,\hat{1})\right) = \begin{cases} 0, & \text{if } \vee S<\pi \\ -1, & \text{if } \vee S=\pi \end{cases}$$
6. The inclusion-exclusion formula gives
$$\mu(A(\sigma,\pi)) = (-1)^{|\pi|-|\sigma|}|\,\mathrm{NE}(\sigma,\pi)| + \sum_{\substack{S\subseteq \mathrm{RepEmb}(\sigma,\pi)\\ \vee S=\pi}} (-1)^{|S|}$$
7. Steps 1, 2 and 6 give Equation (1)
8. $A(\sigma,\pi)$ and $A^*(\lambda,\pi)$, $\lambda\in[\sigma,\pi]$, can be shelled using a recursive atom ordering if $[\sigma,\pi]$ has no zero split subintervals and this implies $[\sigma,\pi]$ is shellable
9. Equation (3) is obtained by counting the decreasing chains in $A^*(\lambda,\pi)$

Further Work

- What is the effect of zero split subintervals on the Möbius function?
- What classes of intervals have no zero split subintervals?